数理解析研究所講究録
第 1893 巻 2014 年 136-142

Order of operators determined by operator mean

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1 Introduction

This is a joint work with Prof. M. Uchiyama.

Let J be an open interval of $\mathbb{R}$. We define H_n, $H_n(J)$ and H_n^+ as follows:

$$H_n = \{A \in \mathbb{M}_n(\mathbb{C}) \mid A = A^*\}$$
$$H_n(J) = \{A \in H_n \mid \mathrm{Sp}(A) \subset J\}$$
$$H_n^+ = H_n([0, \infty)).$$

We call f an operator monotone function on J if we have $f(A) \leq f(B)$ for any $A, B \in H_n(J)$ with $A \leq B$. The following functions are well known as typical examples of operator monotone functions:

$$f(t) = t^p \quad (0 \leq p < 1) \qquad \text{on } J = [0, \infty),$$
$$f(t) = \frac{at+b}{ct+d} \quad (a, b, c, d \in \mathbb{R}, ad - bc = 1) \qquad \text{on } J = (-\infty, -d/c) \text{ or } (-d/c, \infty).$$

For the operator monotone function f on J, it does not necessarily follow that

$$A, B \in H_n(J),\ f(A) \leq f(B) \Rightarrow A \leq B.$$

So we consider the following condition for $C \in H_n(J)$ and $A, B \in H_n$:

$$f(C + tA) \leq f(C + tB) \qquad \text{for any } 0 < t < \epsilon, \tag{*}$$

where ϵ is a sufficiently small positive number. One of our problems is to determine the condition for f or for C, which deduces $A \leq B$ from the condition$(*)$.

By Kubo-Ando theory [5], it is known that an operator mean σ is related to the operator monotone function f on $[0, \infty)$ with $f(1) = 1$, that is, for $A, B \in H_n((0, \infty))$, the operator mean $A\sigma B$ of A and B is represented as the following form:

$$A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

So we can naturally consider the following condition for $X, Y \in H_n((0, \infty))$ and $A, B \in H_n$ which is similar to above problem:

$$Y\sigma(tA + X) \leq Y\sigma(tB + X) \qquad \text{for any } 0 < t < \epsilon, \tag{**}$$

where ϵ is a sufficiently small positive number. Our results is as follows:

Theorem 1. *The condition (**) implies $A \le B$ is equivalent to that X is a scalar multiple of Y or the operator monotone function f associated with σ has the form* $f(t) = \dfrac{at+b}{ct+d}$.

2 Outline of Proof

We show the following:
Fact 1. When $X = cY$ for some positive scalar c, (**) implies $A \le B$.
Fact 2. When the operator monotone function f has the following form:

$$f(t) = \frac{at+b}{ct+d} \qquad a, b, c, d \in \mathbb{R},\ ad - bc > 0,$$

(**) implies $A \le B$.
Fact 3. When X is not scalar multiple of Y and f does not have the form $f(t) = \dfrac{at+b}{ct+d}$, then there exist positive operators A and B such that $A \not\le B$ and they satisfy the condition (**) for X, Y and f.

Combining these facts, we can get Theorem 1. So we will explain these facts.

Let f be an operator monotone function on J. For $A \in H_n(J)$, we denote the Fréchet derivative of f at A by $Df(A)$, that is,

$$\lim_{\|H\| \to 0} \frac{\|f(A+H) - f(A) - D(f)(A)\|}{\|H\|} = 0.$$

We remark $Df(A)$ a bounded real linear operator on H_n. We also denote the directional derivative of f at A in the direction B by $Df(A)(B)$, that is,

$$Df(A)(B) = \left.\frac{d}{dt}\right|_{t=0} f(A+tB).$$

We choose some unitary U such that

$$\Lambda = U^*AU = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Then it is known that

$$Df(A)(B) = U(f^{[1]}(\Lambda) \circ (U^*BU))U^*,$$

where $f^{[1]}(\Lambda) = (f^{[1]}(\lambda_i, \lambda_j))$,

$$f^{[1]}(\lambda_i, \lambda_j) = \begin{cases} \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} & \lambda_i \ne \lambda_j \\ f'(\lambda_i) & \lambda_i = \lambda_j \end{cases}$$

and the notation $\circ$ means Schur product of matrices.

Since f is operator monotone, $f^{[1]}(\Lambda)$ becomes positive. When $A = cI$,

$$f^{[1]}(cI) = \begin{pmatrix} f'(c) & \cdots & f'(c) \\ \vdots & \ddots & \vdots \\ f'(c) & \cdots & f'(c) \end{pmatrix}$$

is positive and of rank 1. It is also known that the operator monotone function f has the form

$$f(t) = \frac{at+b}{ct+d},$$

if $f^{[1]}(\Lambda)$ is of rank 1 for some $\Lambda \neq cI$ (see [3]).

The following proposition is a key idea of this paper:

Proposition 2. *For $A = (a_{ij}) \in H_n^+$, we consider the map $S_A : H_n \ni B \mapsto A \circ B \in H_n$. Then the following are equivalent:*

(1) For $B \in H_n$, $S_A(B) \geq 0 \Rightarrow B \geq 0$.

(2) A is **of strict rank 1**, *that is, there exists $\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{pmatrix}$ such that $A = \gamma^* \gamma$ and $\gamma_1 \gamma_2 \cdots \gamma_n \neq 0$.*

(3) $S_A(H_n^+) = H_n^+$.

(4) For any k, l $(1 \leq k, l \leq n)$, $a_{kk} > 0$ and $a_{kk}a_{ll} - a_{kl}a_{lk} = 0$.

We can prove (1) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). This proof has been written in [6]. Here we give only the part (1) $\Rightarrow$ (4) $\Rightarrow$ (2), because the rest part of proof is not so difficult.

Proof. (1) $\Rightarrow$ (4) When $a_{kk} = 0$, we define $B = (b_{ij})$ as follows:

$$b_{ij} = \begin{cases} -1 & \text{if } (i,j) = (k,k) \\ 0 & \text{otherwise} \end{cases}.$$

Since $B \not\geq 0$ and $S_A(B) = A \circ B = 0 \geq 0$, this contradicts to the assumption. So $a_{kk} > 0$ for all k.

The positivity of A implies that

$$\begin{pmatrix} a_{kk} & a_{kl} \\ a_{lk} & a_{ll} \end{pmatrix} \geq 0,$$

in particular, $a_{kk}a_{ll} - a_{kl}a_{lk} \geq 0$. We assume that $a_{kk}a_{ll} - a_{kl}a_{lk} > 0$. We define $B = (b_{ij})$ as follows:

$$b_{ij} = \begin{cases} \dfrac{|a_{kl}|}{a_{kk}} & \text{if } (i,j) = (k,k) \\ \dfrac{|a_{kl}|}{a_{ll}} & \text{if } (i,j) = (l,l) \\ 1 & \text{if } (i,j) = (k,l) \text{ or } (l,k) \\ 0 & \text{otherwise} \end{cases}.$$

Since $|a_{kl}|^2 = a_{kl}a_{lk} < a_{kk}a_{ll}$, we have $B \ngeq 0$. But we have

$$(A \circ B)_{ij} = \begin{cases} |a_{kl}| & \text{if } (i,j) = (k,k) \text{ or } (l,l) \\ a_{kl} & \text{if } (i,j) = (k,l) \\ a_{lk} & \text{if } (i,j) = (l,k) \\ 0 & \text{otherwise} \end{cases},$$

and $A \circ B \geq 0$. This contradicts to the assumption. So we can get the following:

$$a_{kk},\ a_{ll} > 0,\ a_{kk}a_{ll} = a_{kl}a_{lk}(= |a_{kl}|^2).$$

(4) $\Rightarrow$ (2) Define $r_k > 0$ $(k = 1, 2, \ldots, n)$ by the following relation:

$$a_{kk} = r_k^2.$$

Then, for any k and l, we can choose $\theta(k,l) \in \mathbb{R}$ such that

$$a_{kl} = r_k r_l e^{i\theta(k,l)},$$

and we may assume that the following relation:

$$e^{i\theta(k,l)} = e^{-i\theta(l,k)}, \quad e^{i\theta(k,k)} = 1.$$

If we show the relation

$$e^{i\theta(k,l)}e^{i\theta(l,m)} = e^{i\theta(k,m)}$$

for any k, l and m, then we can see that A is of strict rank 1 as follows:

$$\begin{aligned}
&\begin{pmatrix} r_1 \\ r_2 e^{-i\theta(1,2)} \\ \vdots \\ r_n e^{-i\theta(1,n)} \end{pmatrix} \begin{pmatrix} r_1 & r_2 e^{i\theta(1,2)} & \cdots & r_n e^{i\theta(1,n)} \end{pmatrix} \\
&= \begin{pmatrix} r_1 \\ r_2 e^{i\theta(2,1)} \\ \vdots \\ r_n e^{i\theta(n,1)} \end{pmatrix} \begin{pmatrix} r_1 & r_2 e^{i\theta(1,2)} & \cdots & r_n e^{i\theta(1,n)} \end{pmatrix} \\
&= \begin{pmatrix} r_1^2 & r_1 r_2 e^{i\theta(1,2)} & \cdots & r_1 r_n e^{i\theta(1,n)} \\ r_2 r_1 e^{i\theta(2,1)} & r_2^2 e^{i\theta(2,1)} e^{i\theta(1,2)} & \cdots & r_2 r_n e^{i\theta(2,1)} e^{i\theta(1,n)} \\ \vdots & \vdots & \ddots & \vdots \\ r_n r_1 e^{i\theta(n,1)} & r_n r_2 e^{i\theta(n,1)} e^{i\theta(1,2)} & \cdots & r_n^2 e^{i\theta(n,1)} e^{i\theta(1,n)} \end{pmatrix} \\
&= \begin{pmatrix} r_1^2 & r_1 r_2 e^{i\theta(1,2)} & \cdots & r_1 r_n e^{i\theta(1,n)} \\ r_2 r_1 e^{i\theta(2,1)} & r_2^2 & \cdots & r_2 r_n e^{i\theta(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ r_n r_1 e^{i\theta(n,1)} & r_n r_2 e^{i\theta(n,2)} & \cdots & r_n^2 \end{pmatrix} = A.
\end{aligned}$$

It suffices to show the relation $e^{i\theta(k,l)}e^{i\theta(l,m)} = e^{i\theta(k,m)}$ in the case of each two of k, l, m are different. By the positivity of A, we have

$$\begin{pmatrix} a_{kk} & a_{kl} & a_{km} \\ a_{lk} & a_{ll} & a_{lm} \\ a_{mk} & a_{ml} & a_{mm} \end{pmatrix} \geq 0.$$

Since

$$\begin{pmatrix} a_{kk} & a_{kl} & a_{km} \\ a_{lk} & a_{ll} & a_{lm} \\ a_{mk} & a_{ml} & a_{mm} \end{pmatrix} = \begin{pmatrix} r_k^2 & r_k r_l e^{i\theta(k,l)} & r_k r_m e^{i\theta(k,m)} \\ r_l r_k e^{i\theta(l,k)} & r_l^2 & r_l r_m e^{i\theta(l,m)} \\ r_m r_k e^{i\theta(m,k)} & r_m r_l e^{i\theta(m,l)} & r_m^2 \end{pmatrix}$$
$$= \begin{pmatrix} r_k e^{i\theta(k,l)} & & \\ & r_l & \\ & & r_m e^{i\theta(m,l)} \end{pmatrix} \begin{pmatrix} 1 & 1 & \alpha \\ 1 & 1 & 1 \\ \bar{\alpha} & 1 & 1 \end{pmatrix} \begin{pmatrix} r_k e^{i\theta(l,k)} & & \\ & r_l & \\ & & r_m e^{i\theta(l,m)} \end{pmatrix}$$

and

$$\alpha = e^{-i\theta(k,l)}e^{-i\theta(l,m)}e^{i\theta(k,m)},$$

we have

$$\begin{pmatrix} 1 & 1 & \alpha \\ 1 & 1 & 1 \\ \bar{\alpha} & 1 & 1 \end{pmatrix} \geq 0.$$

Remarking that $|\alpha| = 1$ and

$$0 \leq \left\langle \begin{pmatrix} 1 & 1 & \alpha \\ 1 & 1 & 1 \\ \bar{\alpha} & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\rangle = \alpha + \bar{\alpha} - 2,$$

we can get $\alpha = 1$. So we have the desired relation. $\square$

We now consider the condition, for $C \in H_n(J)$ and $A, B \in H_n$:

$$f(C + tA) \leq f(C + tB) \qquad \text{for any } 0 < t < \epsilon. \tag{*}$$

Since

$$\frac{f(C + tA) - f(C)}{t} \leq \frac{f(C + tB) - f(C)}{t},$$

we have $Df(C)(A) \leq Df(C)(B)$, i.e., $Df(C)(B - A) \geq 0$. As stated above $f^{[1]}(C)$ is of strict rank 1 when $C = cI$ or $f(t)$ has the form $(at + b)/(ct + d)$. Using the property (1) in Proposition 2, we have the following:

Fact 1'. When $C = cI$ for some scalar in J, (*) implies $A \leq B$.
Fact 2'. When the operator monotone function f on J has the following form:

$$f(t) = \frac{at + b}{ct + d} \quad a, b, c, d \in \mathbb{R},\ ad - bc > 0,$$

(*) implies $A \leq B$.

When f does not have the form $(at+b)/(ct+d)$, $f^{[1]}(\Lambda)$ is not of rank 1 for $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ $(\lambda \neq \mu \in J)$. This means $f'(\lambda)f'(\mu) > f^{[1]}(\lambda,\mu)^2$. So we choose $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \in H_2$ with $h_{11}, h_{22} > 0$ and

$$h_{11}h_{22} < |h_{12}|^2 < \frac{f'(\lambda)f'(\mu)}{f^{[1]}(\lambda,\mu)^2} h_{11}h_{22}.$$

Then $H \not\geq 0$ and $Df(\Lambda)(H) = f^{[1]}(\Lambda) \circ H > 0$. Let $A, B \geq 0$ with $H = B - A$. Since

$$\begin{aligned} 0 < Df(\Lambda)(H) &= Df(\Lambda)(B) - Df(\Lambda)(A) \\ &= \lim_{t \to 0} \left(\frac{f(tB+\Lambda) - f(\Lambda)}{t} - \frac{f(tA+\Lambda) - f(\Lambda)}{t} \right) \\ &= \lim_{t \to 0} \frac{f(tB+\Lambda) - f(tA+\Lambda)}{t}, \end{aligned}$$

there exists $\epsilon > 0$ such that

$$f(tB+\Lambda) - f(tA+\Lambda) \geq 0$$

for $0 < t < \epsilon$. In the case, $A \not\leq B$ because $H \not\geq 0$.

Using the embedding

$$H_2 \ni \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & x_{12} & 0 & \cdots & 0 \\ x_{21} & x_{22} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in H_n,$$

we can prove the following:

Fact 3'. When C is not scalar operator in $H_n(J)$ and f does not have the form $f(t) = \dfrac{at+b}{ct+d}$, then there exist positive operators A and B such that $A \not\leq B$ and they satisfy the condition (*).

Using the relation of an operator monotone function f on $(0,\infty)$ with $f(1) = 1$ and the operator mean σ related with f, i.e.,

$$A\sigma B = B^{1/2} f(A^{-1/2}BA^{-1/2})B^{1/2},$$

we can prove **Fact** i from **Fact** i' $(i = 1, 2, 3)$.

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